MATH 270 Multivariable Calculus

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Multivariable Calculus

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I. Relevant Quantities with Explanations

A. Trigonometry

This section highlights useful reference information that is used throughout the following sections and subsections.

The following are definitions of trigonometric functions. Right triangle definitions are expressed:

$$sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{1}$$

$$cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{2}$$

$$tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} \tag{3}$$

$$csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}} \tag{4}$$

$$sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}} \tag{5}$$

$$cot(\theta) = \frac{\text{adjacent}}{\text{opposite}} \tag{6}$$

The following define Pythagorean identities:

$$\sin^2(\theta) + \cos^2(\theta) = 1 \tag{7}$$

$$\tan^2(\theta) + 1 = \sec^2(\theta) \tag{8}$$

$$1 + \cot^2(\theta) = \csc^2(\theta) \tag{9}$$

The following define reciprocal identities:

$$\sin(\theta) = \frac{1}{\csc(\theta)} \tag{10}$$

$$\cos(\theta) = \frac{1}{\sec(\theta)} \tag{11}$$

$$\tan(\theta) = \frac{1}{\cot(\theta)} \tag{12}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} \tag{13}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \tag{14}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} \tag{15}$$

If we want to go from degrees to radians, the following conversions are performed where x is an angle in degrees and t is an angle in radians:

$$\frac{\pi}{180} = \frac{t}{x} \Longrightarrow t = \frac{\pi x}{180} \text{ and } x = \frac{180t}{\pi} \tag{16}$$

As we work with the unit circle, for any ordered pair on the unit circle (x, y) we define $\cos\theta = x$ and $\sin\theta = y$. The four quadrants can be defined as follows:

Quadrant	**Range and Trigonometric Signs**
Quadrant I	$0° \leq \theta \leq 90°$ $(\cos\theta > 0, \sin\theta > 0)$
Quadrant II	$90° < \theta \leq 180°$ $(\cos\theta < 0, \sin\theta > 0)$
Quadrant III	$180° < \theta \leq 270°$ $(\cos\theta < 0, \sin\theta < 0)$
Quadrant IV	$270° < \theta \leq 360°$ $(\cos\theta > 0, \sin\theta < 0)$

Here are the key angles in each quadrant arranged in a 2x2 matrix:

Quadrant II	Quadrant I
$120° : \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$30° : \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$135° : \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$45° : \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$150° : \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$60° : \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$180° : (-1, 0)$	$90° : (0, 1)$

Quadrant III	Quadrant IV
$210° : \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	$300° : \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$225° : \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	$315° : \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$240° : \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$330° : \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$270° : (0, -1)$	$360° : (1, 0)$

This structure arranges the angles and their corresponding coordinates in the unit circle in a way that reflects the four quadrants.

B. Algebra

We define a series of useful algebraic properties and formulas I use often in this course. The following are arithmetic operations:

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \tag{17}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \tag{18}$$

$$\frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b} \tag{19}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc} \tag{20}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \tag{21}$$

$$\frac{a-b}{c-d} = \frac{b-a}{d-c} \tag{22}$$

$$\frac{a+b}{c} = \frac{a}{c} = \frac{b}{c} \tag{23}$$

The following are factoring formulas:

$$x^2 + a^2 = (x+a)(x-a) \tag{24}$$

$$x^2 + 2ax + a^2 = (x+a)^2 \tag{25}$$

$$x^2 - 2ax + a^2 = (x-a)^2 \tag{26}$$

$$x^2 + (a+b)x + ab = (x+a)(x+b) \tag{27}$$

$$x^3 + 3ax^2 + 3a^2x + a^3 = (x+a)^3 \tag{28}$$

$$x^3 - 3ax^2 + 3a^2x - a^3 = (x-a)^3 \tag{29}$$

$$x^3 + a^3 = (x+a)(x^2 - ax + a^2) \tag{30}$$

$$x^3 - a^3 = (x-a)(x^2 - ax + a^2) \tag{31}$$

The quadratic formula follows as:

$$ax^2 + bx + c = 0, \quad a \neq 0 \tag{32}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{33}$$

To complete the square (use $2x^2 - 6x - 10 = 0$ as an example):

1. Divide by the coefficient of the x^2

$$x^2 - 3x - 5 = 0 \tag{34}$$

2. Move the constant to the other side

$$x^2 - 3x = 5 \tag{35}$$

3. Take half the coefficient of x, square it and add it to both sides

$$x^2 - 3x + \left(-\frac{3}{2}\right)^2 = 5 + \left(-\frac{3}{2}\right) = 5 + \frac{9}{4} = \frac{29}{4} \tag{36}$$

4. Factor the left side

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4} \tag{37}$$

5. Use the square root property

$$x - \frac{3}{2} = \pm\sqrt{\frac{29}{4}} = \pm\frac{\sqrt{29}}{2} \tag{38}$$

6. Solve for x

$$x = \frac{3}{2} \pm \frac{\sqrt{29}}{2} \tag{39}$$

The slope of a line containing the two points (x_1, y_1) and (x_2, y_2) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}} \tag{40}$$

The equation of a circle is given with radius r and center (h, k):

$$(x - h)^2 + (y - k)^2 = r^2 \tag{41}$$

II. Coordinate Systems

A. Two-and-three-dimensional Coordinate Systems

The two axes (x-axis and y-axis) divide the infinite plane into 4 quadrants. In three axes, (x-axis, y-axis, and z-axis), the 3-dimensional space is divided into 8 octants.

1. If we have a point in a two-dimensional space, the projection of (α, β) to the x-axis is $(x, 0)$.
2. If we have a point in a two-dimensional space, the project of (α, β) to the y-axis is $(0, y)$.
3. If we have a point in a three-dimensional space, the project of (α, β, ζ) to the x-axis is $(x, 0, 0)$.
4. If we have a point in a three-dimensional space, the project of (α, β, ζ) to the y-axis is $(0, y, 0)$.
5. If we have a point in a three-dimensional space, the project of (α, β, ζ) to the z-axis is $(0, 0, z)$.
6. If we have a point in a three-dimensional space, the project of (α, β, ζ) to the xy-axis is $(x, 0, z)$.
7. If we have a point in a three-dimensional space, the project of (α, β, ζ) to the yz-axis is $(0, y, z)$.
8. If we have a point in a three-dimensional space, the project of (α, β, ζ) to the xz-axis is $(x, 0, z)$.

B. 3-D Coordinate System

The distance between points in $\mathbb{R}^2$,$\mathbb{R}^3$, $\mathbb{R}^n$ is given by:

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{42}$$

$$d(P_1, P_2, P_3) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{43}$$

$$d(P_1, P_2, P_3, P_n, \dots) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (n_2 - n_1)^2 + \dots} \tag{44}$$

The equation of a circle is given with radius r and center (h, k):

$$(x - h)^2 + (y - k)^2 = r^2 \tag{45}$$

and the general equation of a sphere is given by:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \tag{46}$$

III. Vectors

A *directed segment*, or *bound vector*, is an ordered pair of points, P and Q denoted by $\overrightarrow{PQ}$ where P is the *initial point* and Q is the *terminal point*. The *magnitude* or *length* of $|\overrightarrow{PQ}|$ is the distance between P and Q. The following properties of the relation $\sim$ (equivalence relation) is given by:

1. The relation is *reflexive*
$$\overrightarrow{PQ} \sim \overrightarrow{PQ} \tag{47}$$
2. The relation is *symmetric*
$$\text{If } \overrightarrow{PQ} \sim \overrightarrow{RS}, \text{then } \overrightarrow{RS} \sim \overrightarrow{PQ} \tag{48}$$
3. The relation is *transitive*
$$\text{If } \overrightarrow{PQ} \sim \overrightarrow{RS}, \text{and } \overrightarrow{RS} \sim \overrightarrow{TU}, \text{then } \overrightarrow{PQ} \sim \overrightarrow{TU} \tag{49}$$

In *component form*, vectors can be described and compared. If a vector v_1 has an initial point $A(x_1, y_1, z_1)$ and a terminal point $B(x_2, y_2, z_2)$, then its component form is $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$. A few additional statements:

$$\overrightarrow{a} + \overrightarrow{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle \tag{50}$$
$$c\overrightarrow{a} = \langle ca_1, ca_2, ca_3\rangle \tag{51}$$

Magnitude of a vector is the length of vector $\|v\|$. The formula is given:

$$\overrightarrow{v}\langle v_n\rangle = \sqrt{v^2{}_n} \tag{52}$$

It is the distance from the endpoint of a vector to the origin (length). This is a *scalar* value which represents the length independent of direction.

A *unit* vector is a vector whose magnitude is 1. In $\mathbb{R}^3$, the unit vector is given:

$$\overrightarrow{a} = \langle a_1, a_2, a_3\rangle = a_1\langle 1, 0, 0\rangle + a_2\langle 0, 1, 0\rangle + a_3\langle 0, 0, 1\rangle = a_1\overrightarrow{i} + a_2\overrightarrow{j} + a_3\overrightarrow{k} \tag{53}$$

If $\overrightarrow{a} \neq \overrightarrow{0}$, then $\|\overrightarrow{a}\| \neq 0$. The associated vector $\overrightarrow{u} = \frac{1}{\|\overrightarrow{a}\|}\|\overrightarrow{a}$ is called the *unit vector* in the direction of $\overrightarrow{a}$. If two vectors point in separate directions, then they are said to be *linearly independent*. If $\overrightarrow{a}$ and $\overrightarrow{b}$ point in the same direction, then it is possible to go from $\overrightarrow{a} \longrightarrow \overrightarrow{b}$ by multiplying $\overrightarrow{a}$ by a constant scalar value and vice versa. If not, then its not possible to make one out of the other as you can never change the direction of the vector.

IV. Vector Applications

Force can be represented as a vector since it has a magnitude and direction. When two or more forces act on an object, the sum of these forces give the *resultant* force which is the force experienced by the object.

$$\overrightarrow{F} = |\overrightarrow{F}|\cos\theta\,\overrightarrow{i} + |\overrightarrow{F}|\sin\theta\,\overrightarrow{j} \tag{54}$$

In an example where a mass of n weight is being suspended from two cables such that each for θ and θ' angles with the horizontal, we know that the two tension vectors sustaining the mass must add up as a resultant vector in the opposite direction of $\overrightarrow{N}$.

$$\overrightarrow{F_1} + \overrightarrow{F_2} = \overrightarrow{N}\langle 0, n\rangle \tag{55}$$

The component form of F_1 and F_2 is given by analyzing each vector and its individual components. Take $\overrightarrow{F_1}$ as an example. Using the $SOH - CAH - TOA$ ratios, we can use the magnitude of the $\overrightarrow{F_1}$ vector and $\sin\theta$ of the θ angle it makes with the horizontal to find the y component. We can use the magnitude of the $\overrightarrow{F_1}$ vector and $\cos\theta$ of the θ angle it makes with the horizontal to find the x component—this will be negative or positive depending on which way you set up your vectors:

$$\overrightarrow{F_1} = -\|\overrightarrow{F_1}\| \cos\theta \overrightarrow{i} + \|\overrightarrow{F_2}\| \sin\theta \overrightarrow{j} \tag{56}$$

. You will eventually finish with two systems of equations which you can solve for.

$$\sum \overrightarrow{F_x} = \|\overrightarrow{F_2}\| \cos 40 - \|\overrightarrow{F_1}\| \cos 55 = 0 \tag{57}$$

$$\sum \overrightarrow{F_{yx}} = \|\overrightarrow{F_1}\| \sin 55 + \|\overrightarrow{F_2}\| \sin 40 = 75 \tag{58}$$

A. Dot Products

The *dot product* involves multiplying corresponding components of vectors and summing the products. The result is a *constant*. The dot product signifies the amount one vector goes in the direction of another.

1. When two vectors are pointing in the same general direction, the dot product is positive.
2. When perpendicular $\perp$, the projection of one onto the other is the zero vector $\overrightarrow{0}$, the dot product is zero.
3. When pointing in generally the opposite direction, their dot product is negative.

It can also be described as:

$$\text{Length of projected } \overrightarrow{w} \times \text{length of } \overrightarrow{v} \tag{59}$$

Order does not matter. This can be performed vice versa and get the same result.

$$\text{Length of projected } \overrightarrow{v} \times \text{length of } \overrightarrow{w} \tag{60}$$

$$\overrightarrow{v}\langle v_1, v_2, v_n\rangle \cdot \overrightarrow{w}\langle w_1, w_2, w_n\rangle \longrightarrow \overrightarrow{v} \cdot \overrightarrow{w} = v_1w_1 + v_2w_2 + v_nw_n \tag{61}$$

The following are properties of dot products:

1. $$\overrightarrow{v} \cdot \overrightarrow{v} = \|\overrightarrow{v}\|^2 \tag{62}$$
2. $$\overrightarrow{v} \cdot \overrightarrow{w} = \overrightarrow{w} \cdot \overrightarrow{v} \tag{63}$$
3. $$\overrightarrow{v} \cdot (\overrightarrow{w} + \overrightarrow{u}) = \overrightarrow{v} \cdot \overrightarrow{w} + \overrightarrow{v} \cdot \overrightarrow{u} \tag{64}$$
4. $$(c\overrightarrow{v}) \cdot \overrightarrow{w} = c(\overrightarrow{v} \cdot \overrightarrow{w}) = \overrightarrow{v} \cdot (c\overrightarrow{w}) \tag{65}$$

Geometrically, the *dot product* equates

$$\overrightarrow{v} \cdot \overrightarrow{w} = \|\overrightarrow{v}\|\|\overrightarrow{w}\| \cos\theta \tag{66}$$

where θ is the angle between $\overrightarrow{v}$ and $\overrightarrow{w}$.

B. Parallel and Perpindicular Vectors

If $\overrightarrow{v}$ and $\overrightarrow{w}$ are parallel, then $\overrightarrow{v} = c\overrightarrow{w}$. If $\overrightarrow{v}$ and $\overrightarrow{w}$ are perpendicular (orthogonal), then $\overrightarrow{v} \cdot \overrightarrow{w} = 0$. The dot product between two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ is given by $\overrightarrow{v} \cdot \overrightarrow{w} = \|\overrightarrow{v}\|\|\overrightarrow{w}\| \cos\theta$ where θ is the angle $0 \leq \theta \leq \pi$:

1. Positive if
$$\cos\theta > 0 \iff 0 \leq \theta < \frac{\pi}{2} \tag{67}$$
2. Negative if
$$\cos\theta < 0 \iff \frac{\pi}{2} < \theta \leq \pi \tag{68}$$
3. θ is an acute ($\theta < 90°$) angle if
$$\overrightarrow{v} \cdot \overrightarrow{w} > 0 \tag{69}$$
4. θ is a right angle if
$$\overrightarrow{v} \cdot \overrightarrow{w} = 0 \tag{70}$$
5. θ is an obtuse ($\theta > 90°$) angle if
$$\overrightarrow{v} \cdot \overrightarrow{w} < 0 \tag{71}$$

A three-dimensional vector is an arrow with both length and direction. It is a line segment with a start and end point with the ordered triple $\overrightarrow{v} = \langle v_x, v_y, v_z \rangle$ of real numbers (its components). A unit vector is a vector whose length is 1. It has the same direction as $\overrightarrow{v} = \langle v_x, v_y, v_z \rangle$.

$$\overrightarrow{u} = \frac{1}{\|\overrightarrow{v}\|}\overrightarrow{v} = \langle \frac{v_x}{\|\overrightarrow{v}\|}, \frac{v_y}{\|\overrightarrow{v}\|}, \frac{v_z}{\|\overrightarrow{v}\|} \rangle \tag{72}$$

The *direction angles* of a nonzero vector $\overrightarrow{v}$ are the angles α, β, γ in the interval $[0, \pi]$ that $\overrightarrow{v}$ makes with the x, y, and z axes. The cosines of these direction angles, $\cos\alpha, \cos\beta, \cos\gamma$ are called the *direction cosines* of $\overrightarrow{v}$.

1.
$$\cos\alpha = \frac{\overrightarrow{v} \cdot \overrightarrow{i}}{\|\overrightarrow{v}\|\|\overrightarrow{i}\|} = \frac{v_x}{\|\overrightarrow{v}\|} \tag{73}$$
2.
$$\cos\beta = \frac{v_y}{\|\overrightarrow{v}\|} \tag{74}$$
3.
$$\cos\gamma = \frac{v_z}{\|\overrightarrow{v}\|} \tag{75}$$
4.
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{v_x}{\|\overrightarrow{v}\|}\right)^2 + \left(\frac{v_y}{\|\overrightarrow{v}\|}\right)^2 + \left(\frac{v_z}{\|\overrightarrow{v}\|}\right)^2 = 1 \tag{76}$$

We can write
$$\overrightarrow{v} = \langle v_x, v_y, v_z \rangle = \langle \|\overrightarrow{v}\| \cos\alpha, \|\overrightarrow{v}\| \cos\beta, \|\overrightarrow{v}\| \cos\gamma \rangle = \|\overrightarrow{v}\| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle \tag{77}$$
Therefore
$$\frac{1}{\|\overrightarrow{v}\|}\overrightarrow{v} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle \tag{78}$$
which defines the direction cosines of $\overrightarrow{v}$ as the components of the unit vector in the direction of $\overrightarrow{v}$.